Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel
International Advanced Level

Centre Number | Candidate Number

Time 1 hour 30 minutes

Paper reference **WFM01/01**

Mathematics

International Advanced Subsidiary/Advanced Level
Further Pure Mathematics F1

You must have:
Mathematical Formulae and Statistical Tables (Yellow), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided
  – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets
  – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.
- Good luck with your examination.

Turn over ▶

Pearson

Leave blank

1. (i)
$$f(x) = x^3 + 4x - 6$$

(a) Show that the equation $f(x) = 0$ has a root α in the interval $[1, 1.5]$ **(2)**

(b) Taking 1.5 as a first approximation, apply the Newton Raphson process twice to $f(x)$ to obtain an approximate value of α. Give your answer to 3 decimal places. Show your working clearly. **(4)**

(ii)
$$g(x) = 4x^2 + x - \tan x$$

where x is measured in radians.

The equation $g(x) = 0$ has a single root β in the interval $[1.4, 1.5]$

Use linear interpolation on the values at the end points of this interval to obtain an approximation to β. Give your answer to 3 decimal places. **(4)**

ia. $f(1) = (1)^3 + 4(1) - 6 = -1$

$f(1.5) = (1.5)^3 + 4(1.5) - 6 = \frac{27}{8}$

There is a sign change from negative to positive and $f(x)$ is continuous, $\therefore$ root, α, lies in the interval $[1, 1.5]$.

b. $x_1 = 1.5$

we are looking for 3rd approximation (x_3)

Numerical Sol^ns of Eq^ns:

Newton-Rhapson iteration for solving $f(x)=0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

From Newton-Rhapson iteration: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

Start by finding x_2

Leave blank

DO NOT WRITE IN THIS AREA

Question 1 continued

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

(1) must find $f(x_1)$ by subbing in x_1 (1.5) into $f(x)$

(2) must find $f'(x_1)$ by differentiating $f(x)$ and subbing in x_1 (1.5) into $f'(x)$.

(1) $f(1.5) = \frac{27}{8}$ used from part (a)

(2) $f(x) = x^3 + 4x - 6$

differentiate w.r.t. x

$f'(x) = 3x^2 + 4$

$f'(1.5) = 3(1.5)^2 + 4 = \frac{43}{4}$

$x_2 = 1.5 - \frac{\frac{27}{8}}{\frac{43}{4}}$

$x_2 = \frac{51}{43}$

Now can use x_2 to find x_3.

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

(1) must find $f(x_2)$ by subbing in x_2 $(\frac{51}{43})$ into $f(x)$

(2) must find $f'(x_2)$ by differentiating $f(x)$ and subbing in x_2 $(\frac{51}{43})$ into $f'(x)$.

(1) $f(\frac{51}{43}) = (\frac{51}{43})^3 + 4(\frac{51}{43}) - 6 = 0.4126051794$

(2) $f(x) = x^3 + 4x - 6$

differentiate w.r.t. x

$f'(x) = 3x^2 + 4$

Leave blank

Question 1 continued

$f'\left(\frac{51}{43}\right) = 3\left(\frac{51}{43}\right)^2 + 4 = \frac{15199}{1849}$

$$x_3 = \left(\frac{51}{43}\right) - \frac{0.412605179 4}{\left(\frac{15199}{1849}\right)}$$

$x_3 = 1.135851961$

$\boxed{\alpha = 1.136 \ (3\text{d.p.})}$

ii. Linear Interpolation:

$$\alpha = \frac{af(b) - bf(a)}{f(b) - f(a)} \quad \text{for interval} \quad [a,b]$$

where α is the root

$a = 1.4$, $b = 1.5$

$$\beta = \frac{1.4\,g(1.5) - 1.5\,g(1.4)}{g(1.5) - g(1.4)}$$

$g(1.4) = 4(1.4)^2 + (1.4) - \tan(1.4^c) = 3.442116285$

$g(1.5) = 4(1.5)^2 + (1.5) - \tan(1.5^c) = -3.601419947$

$$\beta = \frac{1.4(-3.601419947) - 1.5(3.442116285)}{(-3.601419947) - (3.442116285)}$$

$\beta = 1.44886915$

$\boxed{\beta = 1.449 \ (3\text{ d.p.})}$

2. The complex numbers z_1, z_2 and z_3 are given by

$$z_1 = 2 - \mathrm{i} \qquad z_2 = p - \mathrm{i} \qquad z_3 = p + \mathrm{i}$$

where p is a real number.

(a) Find $\dfrac{z_2 z_3}{z_1}$ in the form $a + b\mathrm{i}$ where a and b are real. Give your answer in its simplest form in terms of p.

(3)

Given that $\left|\dfrac{z_2 z_3}{z_1}\right| = 2\sqrt{5}$

(b) find the possible values of p.

(4)

a. $\dfrac{z_2 z_3}{z_1} = \dfrac{(p-i) \times (p+i)}{2-i}$

$= \dfrac{p^2 - i^2}{2-i}$

$= \dfrac{p^2 - (-1)}{2-i}$

$= \dfrac{p^2+1}{2-i}$

Must now rewrite $\dfrac{p^2+1}{2-i}$ in form $a+bi$:

multiply top and bottom by complex conjugate of denominator

$\dfrac{p^2+1}{2-i} \times \dfrac{2+i}{2+i} = \dfrac{(p^2+1)(2+i)}{(2-i)(2+i)}$

$= \dfrac{2p^2 + p^2 i + 2 + i}{4 - 2i + 2i - i^2}$

$= \dfrac{(2p^2+2) + (p^2+1)i}{4 - i^2}$

$= \dfrac{(2p^2+2) + (p^2+1)i}{4-(-1)}$

Question 2 continued

$$\frac{(2p^2+2)+(p^2+1)i}{5}$$

$$= \frac{2p^2+2}{5} + \frac{p^2+1}{5}i$$

$$\boxed{\frac{z_2 z_3}{z_1} = \frac{2p^2+2}{5} + \frac{p^2+1}{5}i \qquad a = \frac{2p^2+2}{5}, \quad b = \frac{p^2+1}{5}}$$

b. $\left|\frac{z_2 z_3}{z_1}\right| = 2\sqrt{5}$

$$\left|\frac{2p^2+2}{5} + \frac{p^2+1}{5}i\right| = 2\sqrt{5}$$

$$\sqrt{\left(\frac{2p^2+2}{5}\right)^2 + \left(\frac{p^2+1}{5}\right)^2} = 2\sqrt{5}$$

square both sides

$$\left(\cancel{\sqrt{}}\left(\frac{2p^2+2}{5}\right)^2 + \left(\frac{p^2+1}{5}\right)^2\right)^{\cancel{2}} = (2\sqrt{5})^2$$

$$\left(\frac{2p^2+2}{5}\right)^2 + \left(\frac{p^2+1}{5}\right)^2 = 20$$

$$\frac{(2p^2+2)^2}{(5)^2} + \frac{(p^2+1)^2}{(5)^2} = 20$$

$$\frac{(2p^2+2)^2 + (p^2+1)^2}{(5)^2} = 20$$

$$(2p^2+2)^2 + (p^2+1)^2 = 20 \times (5)^2$$

$$(2(p^2+1))^2 + (p^2+1)^2 = 500$$

$$2^2(p^2+1)^2 + (p^2+1)^2 = 500$$

$$4(p^2+1)^2 + (p^2+1)^2 = 500$$

$$5(p^2+1)^2 = 500$$

$$(p^2+1)^2 = \frac{500}{5}$$

Leave blank

Question 2 continued

$(p^2+1)^2 = 100$

$p^2+1 = \pm\sqrt{100}$

$p^2+1 = \pm 10$

$p^2+1 = 10$ or $p^2+1 = -10$

$p^2 = 9$ $\quad$ $p^2 = -11$

$p = \pm\sqrt{9}$ $\quad$ $p = \pm\sqrt{-11}$

$p = \pm 3$ $\quad$ $p = \pm\sqrt{11}\,i$

Q states, however, P is a real number $\therefore$ $p = \pm 3$ only.

$\boxed{p = \pm 3}$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

3. The triangle T has vertices $A(2,1)$, $B(2,3)$ and $C(0,1)$.

The triangle T' is the image of T under the transformation represented by the matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(a) Find the coordinates of the vertices of T' **(2)**

(b) Describe fully the transformation represented by **P** **(2)**

The 2×2 matrix **Q** represents a reflection in the x-axis and the 2×2 matrix **R** represents a rotation through 90° anticlockwise about the origin.

(c) Write down the matrix **Q** and the matrix **R** **(2)**

(d) Find the matrix **RQ** **(2)**

(e) Give a full geometrical description of the single transformation represented by the answer to part (d). **(2)**

a. $$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 2 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \begin{matrix} A' & B' & C' \end{matrix} \\ \begin{bmatrix} (0)(2)+(1)(1) & (0)(2)+(1)(3) & (0)(0)+(1)(1) \\ (-1)(2)+(0)(1) & (-1)(2)+(0)(3) & (-1)(0)+(0)(1) \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \begin{matrix} A' & B' & C' \end{matrix} \\ \begin{bmatrix} 1 & 3 & 1 \\ -2 & -2 & 0 \end{bmatrix} \end{array}$$

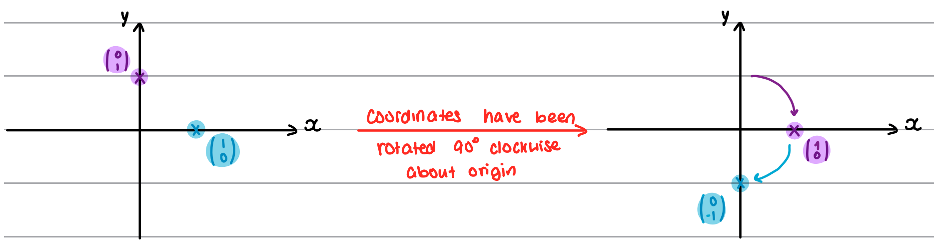
$\boxed{\text{New Vertices : } (1,-2), (3,-2), (1,0)}$

b. Start by drawing a coordinate axis and mark coordinates $\binom{1}{0}$ and $\binom{0}{1}$ then mark the coordinates of new matrix

identity matrix I

matrix P

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{transformation}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$



$\boxed{\text{P represents a rotation 90° clockwise about origin.}}$

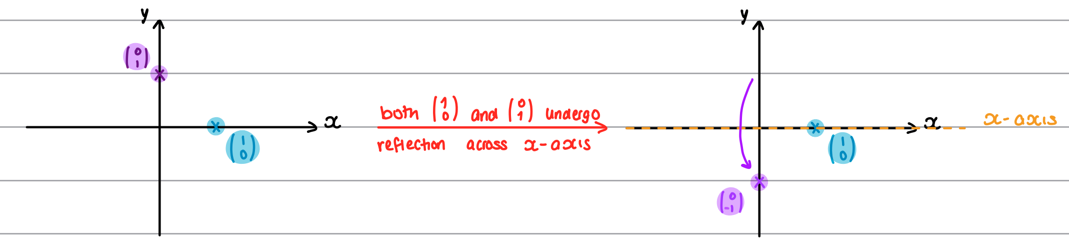
Leave blank

Leave blank

Question 3 continued

c. Start by drawing a coordinate axis and mark coordinates $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$

then transform both of these coordinates by transformation (reflection in x-axis)



identity matrix I

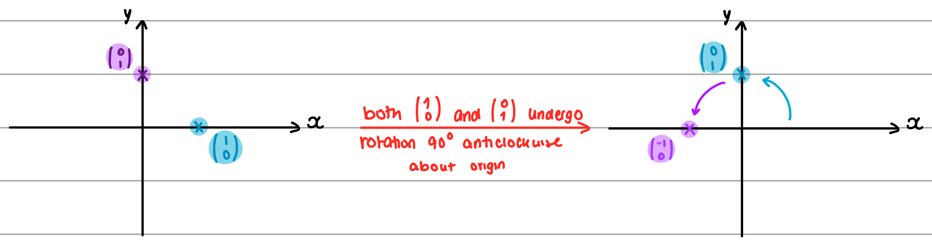
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{new matrix represents transformation } Q} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

rewrite matrix but now replace coordinates with their transformations in this specific order

$$\boxed{Q : \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

Start by drawing a coordinate axis and mark coordinates $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$

then transform both of these coordinates by transformation (rotation $90°$ anticlockwise about origin)



identity matrix I

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{new matrix represents transformation } R} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

rewrite matrix but now replace coordinates with their transformations in this specific order

$$\boxed{R : \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$$

d. $RQ = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} (0)(1)+(-1)(0) & (0)(0)+(-1)(-1) \\ (1)(1)+(0)(0) & (1)(0)+(0)(-1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\boxed{RQ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

DO NOT WRITE IN THIS AREA

Leave blank

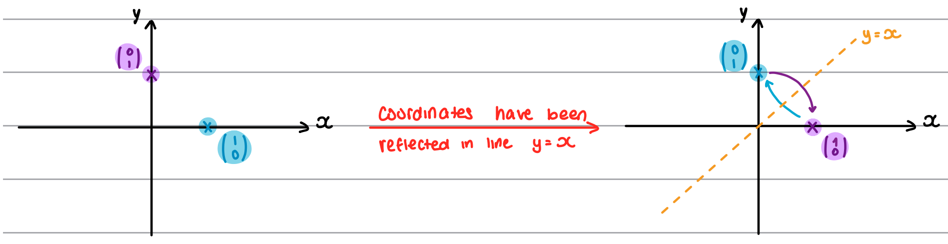
Question 3 continued

e. Start by drawing a coordinate axis and mark coordinates $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$

then mark the coordinates of new matrix

identity matrix I $\qquad\qquad\qquad\qquad\qquad\qquad$ matrix P

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \xrightarrow{\text{transformation}} \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$



$\boxed{\text{RQ represents a reflection in line } y = x}$

Leave blank

4. A rectangular hyperbola H has equation $xy = 25$

The point $P\left(5t, \frac{5}{t}\right)$, $t \neq 0$, is a general point on H.

(a) Show that the equation of the tangent to H at P is $t^2y + x = 10t$

(4)

The distinct points Q and R lie on H. The tangent to H at the point Q and the tangent to H at the point R meet at the point $(15, -5)$.

(b) Find the coordinates of the points Q and R.

(4)

a. Differentiate rectangular hyperbola eq^n w.r.t. x:

$xy = 25$

$y = \frac{25}{x} = 25x^{-1}$

$\frac{dy}{dx} = -25x^{-2} = \frac{-25}{x^2}$

Sub in point $P(5t, \frac{5}{t})$:

$\left.\frac{dy}{dx}\right|_{(5t, \frac{5}{t})} \frac{-25}{(5t)^2} = \frac{-25}{25t^2} = \frac{-1}{t^2}$

$M_{tangent} = -\frac{1}{t^2}$

set up line eq^n $y - y_1 = m(x - x_1)$ with $m = \frac{-1}{t^2}$

$(x_1, y_1) = (5t, \frac{5}{t})$

$y - \frac{5}{t} = -\frac{1}{t^2}(x - 5t)$

$\times t^2$ on both sides

$t^2\left(y - \frac{5}{t}\right) = \cancel{t^2}\left(-\frac{1}{\cancel{t^2}}(x - 5t)\right)$

$t^2y - 5t = -(x - 5t)$

$t^2y - 5t = -x + 5t$

$\boxed{t^2y + x = 10t \text{ // shown}}$

b. Sub in $(15, -5)$ into tangent eq^n from part (a).

$t^2(-5) + (15) = 10t$

$-5t^2 + 15 = 10t$

$5t^2 + 10t - 15 = 0$ $\div 5$

$t^2 + 2t - 3 = 0$

$(t - 1)(t + 3) = 0$

$t = 1$ or $t = -3$

Leave blank

Question 4 continued

Sub $t=1$ and $t=-3$ into point $P\left(5t, \frac{5}{t}\right)$ to find Q and R.

$t=1 \Rightarrow \left(5(1), \frac{5}{(1)}\right) = (5,5)$

$t=-3 \Rightarrow \left(5(-3), \frac{5}{(-3)}\right) = \left(-15, -\frac{5}{3}\right)$

$\boxed{\begin{array}{l} Q: \left(-15, -\frac{5}{3}\right) \\ R: (5,5) \end{array}}$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Leave blank

5. $$f(x) = (9x^2 + d)\left(x^2 - 8x + (10d + 1)\right)$$

where d is a positive constant.

(a) Find the four roots of $f(x)$ giving your answers in terms of d.

(3)

Given $d = 4$

(b) Express these four roots in the form $a + ib$, where $a, b \in \mathbb{R}$.

(2)

(c) Show these four roots on a single Argand diagram.

(2)

a. set $f(x) = 0$ to find roots

$9x^2 + d = 0$

$9x^2 = -d$

$x^2 = -\frac{d}{9}$

$x = \pm\sqrt{-\frac{d}{9}} = \pm\sqrt{(-1)\left(\frac{d}{9}\right)}$

$= \pm\sqrt{\frac{d}{9}}\sqrt{-1}$

$= \pm\frac{\sqrt{d}}{\sqrt{9}}i$

$= \pm\frac{\sqrt{d}}{3}i$

$x^2 - 8x + (10d + 1) = 0$

$(x-4)^2 - 16 + 10d + 1 = 0$

$(x-4)^2 + 10d - 15 = 0$

$(x-4)^2 = 15 - 10d$

$x - 4 = \pm\sqrt{15 - 10d}$

$x = 4 \pm \sqrt{15 - 10d}$

solve for x by completing the square

$\boxed{x = \pm\frac{i\sqrt{d}}{3},\ 4 \pm \sqrt{15 - 10d}}$

b. substitute $d = 4$

$x = \frac{\pm i\sqrt{4}}{3},\ 4 \pm \sqrt{15 - 10(4)}$

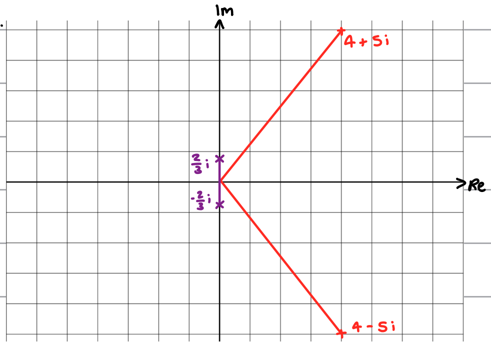
$= \frac{\pm 2i}{3},\ 4 \pm \sqrt{-25}$

$= \frac{\pm 2i}{3},\ 4 \pm 5i$

$\boxed{x = \pm\frac{2i}{3},\ \pm 5 + 4i}$

Question 5 continued

c.



Leave blank

P 6 6 0 6 0 A 0 1 9 3 2

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Leave blank

6. The parabola C has Cartesian equation $y^2 = 8x$

The point $P(2p^2, 4p)$ and the point $Q(2q^2, 4q)$, where $p, q \neq 0$, $p \neq q$, are points on C.

(a) Show that an equation of the normal to C at P is

$$y + px = 2p^3 + 4p$$

(5)

(b) Write down an equation of the normal to C at Q

(1)

The normal to C at P and the normal to C at Q meet at the point N

(c) Show that N has coordinates

$$\left(2(p^2 + pq + q^2 + 2), -2pq(p + q)\right)$$

(5)

The line ON, where O is the origin, is perpendicular to the line PQ

(d) Find the value of $(p + q)^2 - 3pq$

(5)

a. Differentiate parabola eq^n w.r.t. x:

$y^2 = 8x$

$2y\left(\frac{dy}{dx}\right) = 8$

$\frac{dy}{dx} = \frac{8}{2y} = \frac{4}{y}$

Sub in point $P(2p^2, 4p)$:

$\left.\frac{dy}{dx}\right|_{(2p^2, 4p)} \quad \frac{4}{(4p)} = \frac{1}{p}$

$M_{tangent} = \frac{1}{p}$

$M_{normal} = -p$

Set up line eq^n $y - y_1 = m(x - x_1)$ with $m = -p$

$(x_1, y_1) = (2p^2, 4p)$

$y - 4p = -p(x - 2p^2)$

$y - 4p = -px + 2p^3$

$\boxed{y + px = 2p^3 + 4p_{\,//} \text{ shown}}$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question 6 continued

b. replace all the p's with q's.

$\boxed{y + qx = 2q^3 + 4q}$

c. Solve $y+px = 2p^3+4p$ and $y+qx = 2q^3+4q$ simultaneously

(1) $y+px = 2p^3 + 4p$

(2) $y+qx = 2q^3 + 4q$

(1) - (2):
$$y+px = 2p^3 + 4p$$
$$y+qx = 2q^3 + 4q \quad \ominus$$
$$px - qx = 2p^3+4p-2q^3-4q$$
$$(p-q)x = 2p^3 - 2q^3 + 4p - 4q$$
$$(p-q)x = 2[(p^3 - q^3) + 2(p-q)]$$
$$\cancel{(p-q)}x = 2[\cancel{(p-q)}(p^2+pq+q^2) + 2\cancel{(p-q)}]$$

cancel out $(p-q)$ since appears in every term

$$\therefore x = 2(p^2 + pq + q^2 + 2)$$

Now sub into either (1) or (2) to get y-coord.

$$y + p[2(p^2+pq+q^2+2)] = 2p^3 + 4p$$
$$y + \cancel{2p^3} + 2p^2q + 2pq^2 + 4p = \cancel{2p^3} + 4p$$
$$y = -2p^2q - 2pq^2$$
$$y = -2pq(p+q)$$

$\boxed{\therefore \text{coord. } N: \left(2(p^2+pq+q^2+2),\ -2pq(p+q)\right)}$

c. Start by finding gradient of PQ:

$$M_{PQ} = \frac{4q-4p}{2q^2-2p^2} = \frac{4(q-p)}{2(q^2-p^2)} = \frac{\cancel{4}^{2}\cancel{(q-p)}}{\cancel{2}(q+p)\cancel{(q-p)}} = \frac{2}{(p+q)}$$

find gradient of ON:

$$M_{ON} = \frac{-2pq(p+q) - 0}{2(p^2+pq+q^2+2) - 0} = \frac{-2pq(p+q)}{2(p^2+pq+q^2+2)}$$

Leave blank

Question 6 continued

Since lines PQ and ON are perpendicular, $m_{PQ} \times m_{ON} = -1$

$$\frac{\cancel{2}}{\cancel{p+q}} \times \frac{-2pq\cancel{(p+q)}}{\cancel{2}(p^2+pq+q^2+2)} = -1$$

$$\frac{-2pq}{p^2+pq+q^2+2} = \frac{-1}{1}$$

cross multiply

$$-2pq(1) = -1(p^2+pq+q^2+2)$$
$$2pq = p^2+pq+q^2+2$$
$$-2 = p^2+q^2-pq$$
$$-2 = (p+q)^2-2pq-pq$$
$$-2 = (p+q)^2-3pq$$

$$\boxed{\therefore (p+q)^2-3pq = -2}$$

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Leave blank

7. (a) Prove by induction that for $n \in \mathbb{N}$

$$\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$$

(5)

(b) Hence show that

$$\sum_{r=1}^{n} (r^2 + 2) = \frac{n}{6}(an^2 + bn + c)$$

where a, b and c are integers to be found.

(4)

(c) Using your answers to part (b), find the value of

$$\sum_{r=10}^{25} (r^2 + 2)$$

(2)

a. (1) Base case: Prove true for $n = 1$

LHS: $\sum_{r=1}^{1} r^2 = (1)^2 = 1$

RHS: $\frac{(1)}{6}(1+1)(2(1)+1) = 1$

LHS = RHS

$\therefore$ true for $n = 1$

(2) Assume true for $n = k$:

$\sum_{r=1}^{k} r^2 = \frac{k}{6}(k+1)(2k+1)$

(3) Prove true for $n = k+1$:

$\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k+1)^2$ ← we are rewriting sum of (k+1) terms as the sum of (k) terms + $(k+1)^{th}$ term - we can now use step (2).

$= \frac{k}{6}(k+1)(2k+1) + (k+1)^2$

$= \frac{1}{6}(k+1)\left[k(2k+1) + 6(k+1)\right]$

$= \frac{1}{6}(k+1)\left[2k^2 + k + 6k + 6\right]$

$= \frac{1}{6}(k+1)\left[2k^2 + 7k + 6\right]$

Thinking ahead, sub in $n = k+1$ into RHS: this is what we are trying to prove using (2):

$\sum_{r=1}^{k+1} r^2 = \frac{(k+1)}{6}(k+1+1)(2(k+1)+1)$

Make a note so we know what we are working towards

P 6 6 0 6 0 A 0 2 6 3 2

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Leave blank

DO NOT WRITE IN THIS AREA

Question 7 continued

$= \frac{1}{6}(k+1)(k+2)(2k+3)$

$= \frac{(k+1)}{6}(k+1+1)(2(k+1)+1)$ $\therefore$ true for $n=1$

(4) Conclusion

If this result is true for $n=k$, then it is true for $n=k+1$. As the result has been shown to be true for $n=1$, then the result is true for all n.

b. $\sum_{r=1}^{n} r^2+2$

split summation into 2

$\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 2$

factor 2 out summation

$\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} 1$

Summations

$\sum_{r=1}^{n} 1 = n$ must learn!

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ in the formulae booklet

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

$= \frac{1}{6}n(n+1)(2n+1) + 2\boxed{n}$

factor out $\frac{1}{6}n$

$= \frac{1}{6}n\left[(n+1)(2n+1) + 12\right]$

$= \frac{1}{6}n\left[2n^2+3n+1+12\right]$

$= \frac{1}{6}n\left[2n^2+3n+13\right]$

$\boxed{\frac{n}{6}(2n^2+3n+13) \quad a=2, b=3, c=13}$

c. $\sum_{r=10}^{25} r^2+2 = \sum_{r=1}^{25} r^2+2 - \sum_{r=1}^{9} r^2+2$

$= \frac{25}{6}\left(2(25)^2+3(25)+13\right) - \frac{9}{6}\left(2(9)^2+3(9)+13\right)$

$= 5575 - 303$

$= 5272$

$\boxed{5272}$

Leave blank

8. Prove by induction that $4^{n+2} + 5^{2n+1}$ is divisible by 21 for all positive integers n.

(6)

(1) Base Case: check true for $n = 1$

$4^{1+2} + 5^{2(1)+1}$

$= 4^3 + 5^3$

$= 189$

$= 9(21) \longleftarrow$ divisible by 21

$\therefore$ true for $n = 1$

(2) Assume true for $n = k$:

$f(k) = 4^{k+2} + 5^{2k+1}$

(3) Inductive Step, Prove true for $n = k+1$:

$f(k+1) = 4^{(k+1)+2} + 5^{2(k+1)+1}$

$= 4^{k+3} + 5^{2k+3}$

$f(k+1) - f(k) = (4^{k+3} + 5^{2k+3}) - (4^{k+2} + 5^{2k+1})$

$= [(4)^1(4^{k+2}) + (5^2)(5^{2k+1})] - (4^{k+2} + 5^{2k+1})$

$= [4(4^{k+2}) + 25(5^{2k+1})] - [4^{k+2} + 5^{2k+1}]$

$= 4(4^{k+2}) + 25(5^{2k+1}) - (4^{k+2}) - (5^{2k+1})$

$= 3(4^{k+2}) + 24(5^{2k+1})$

$= 3[(4^{k+2}) + (5^{2k+1})] + 21(5^{2k+1})$

$= 3f(k) + 21(5^{2k+1})$

$f(k+1) - f(k) = 3f(k) + 21(5^{2k+1})$

$f(k+1) = 4f(k) + 21(5^{2k+1})$

$\therefore f(k+1) = 4f(k) + 21(5^{2k+1})$ $\therefore$ true for $n = k+1$

(4) Conclusion

If this result is true for $n = k$, then it is true for $n = k+1$. As the result has been shown to be true for $n = 1$, then the result is true for all n.

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA